XXIII. *The general Mathematical Laws which regulate and extend Proportion univerfally; or, a Method of comparing Magnitudes of any Kind together, in all the poffible Degrees of Increafe and Decreafe.* By James Glenie, *A. M. and Lieutenant in the Royal Regiment of Artillery.*

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THE doctrine of proportion laid down by EUCLID, and the application of it given by him in his Elements, form the bafis of almoft all the geometrical reafoning made ufe of by mathematicians both ancient and modern. But the reafonings of geometers with regard to proportional magnitudes have feldom been carried beyond the triplicate *ratio*, which is the proportion that fimilar folids have to one another when referred to their homologous linear dimenfions. This boundary, however, comprehends but a very limited portion of univerfal comparifon, and almoft vanifhes into nothing when referred to that endlefs variety of relations, which muft neceffarily take place between geometrical magnitudes, in the infinite poffible degrees of increafe and decreafe. The firft of thefe takes in but a very

a very contracted field of geometrical comparison; whereas the last extends it indefinitely. Within the narrow compass of the first, the ancient geometers performed wonders, and their labours have been pushed still farther by the ingenuity and indefatigable industry of the moderns. But no author, that I have been able to meet with, gives the least hint or information with regard to any general method of expressing geometrically, when any two magnitudes of the same kind are given, what degree of augmentation or diminution any one of these magnitudes must undergo, in order to have to the other any multiplicate or sub-multiplicate *ratio* of these magnitudes in their given state; or any such *ratio* of them as is denoted by fractions or surds; or (to speak still more generally) a *ratio* which has, to the *ratio* of the first-mentioned of these magnitudes to the other, the *ratio* of any two magnitudes whatever of the same but of any kind. Neither have I been able to find that any author has shewn geometrically in a general way, when any number of *ratios* are to be compounded or decompounded with a given *ratio*, how much either of the magnitudes in the given *ratio* is to be augmented or diminished, in order to have to the other a *ratio*, which is equal to the given *ratio*, compounded or decompounded with the other *ratios*. To investigate all these geome-

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trically, and to fix general laws in relation to them, is the object of this paper; which, as it treats of a ſubject as new as it is general, I flatter myſelf, will not prove unacceptable to this learned Society. It would be altogether ſuperfluous for me to mention the great advantages that muſt neceſſarily accrue to mathematics in general, from an accurate inveſtigation of this ſubject, ſince its influence extends more or leſs to every branch of abſtract ſcience, when any *data* can be aſcertained for reaſoning from. I ſhall, in a ſubſequent paper, take an opportunity of ſhewing how, from the theorems afterwards delivered in this, a method of reaſoning with finite magnitudes, geometrically, may be derived, without any conſideration of motion or velocity, applicable to every thing to which fluxions have been applied; and ſhall now proceed to the ſubject of this paper, after premiſing the two following definitions.

DEFINITION I.

Magnitude is that which admits of increaſe or decreaſe.

DEFINITION II.

Quantity is the degree of Magnitude.

By magnitude, beſides extenſion, I mean every thing which admits of more or leſs, or what can be increaſed

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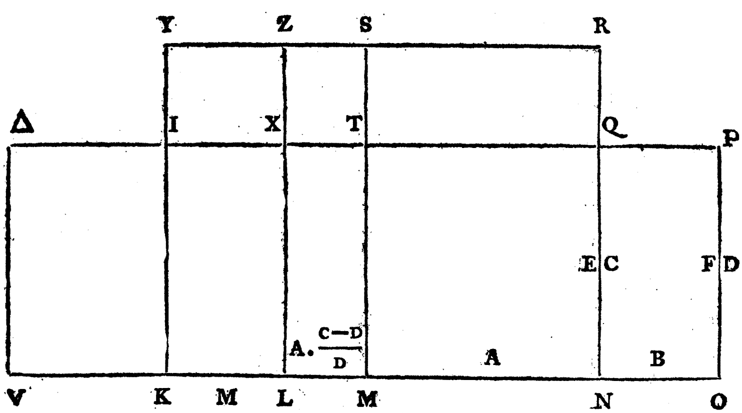
or diminiſhed, ſuch as *ratios*, velocities, powers, &c. As I ſhall frequently, for the ſake of conciſeneſs and conveniency, be obliged to make uſe of particular modes of expreſſing geometrical magnitudes, I here obſerve, once for all, that by ſuch expreſſions as theſe $A.\frac{A}{B}$, $A.\frac{A-B}{B}$, $A.\frac{C-D}{D}$, $A.\frac{\overline{A-B}^2}{B}$, &c. I mean reſpectively a third proportional to B and A; a fourth proportional to B, A, and the difference of A and B; a fourth proportional to D, A, and the difference of C and D; a fourth proportional to B, $A.\frac{A-B}{B}$ and A−B, &c.

To proceed then in the order in which I firſt inveſtigated theſe theorems; let A, B, C, D, E, F, G, H, &c. be any number of magnitudes of the ſame kind, taken two



and two from the firſt; and let MN, NO, NR, OP, reſpectively repreſent A, B, C, D. Let NR, OP, be drawn perpendicularly

pendicularly to VO, or otherwise if in the same angle; and let the rectangles or parallelograms MR, NP, be completed. Let LM be a fourth proportional to OP, MN and NR−OP; and let the rectangle or parallelogram LQ be completed.

Then (14. E. 6.) LT is equal to TR, and consequently LQ to MR. But (23. E. 6.) MR has to NP the *ratio* compounded of the *ratios* of MN to NO and NR to OP. Therefore (1. E. 6.) LN has to NO the *ratio* compounded of the *ratios* of MN to NO and NR to OP. But LN is equal to $MN + MN.\frac{NR-OP}{OP}$, or $A + A.\frac{C-D}{D}$, by construction. Whence it appears, that a magnitude of the same kind with A and B, which has to B the *ratio* compounded of the *ratios* of A to B and C to D, is expressed by $A + A.\frac{C-D}{D}$.

In like manner let E, F, be represented by RN, OP, respectively, and let LK be a fourth proportional to OP, LN, and QR. Then (14. E. 6.) KX is equal to XR or TR and XS together. But since LN hath already been shewn to be equal to $A + A.\frac{C-D}{D}$, LK is a fourth proportional to F, E−F, and $A + A.\frac{C-D}{D}$; that is equal to $A.\frac{E-F}{F} + A.\frac{C-D}{D}.\frac{E-F}{F}$. by construction. Wherefore KN being equal to LK+LN is equal to $A + A.\frac{C-D}{D} + A.\frac{E-F}{F} + A.\frac{C-D}{D}.\frac{E-F}{F}$. And since KQ is equal to LR, KN has to NO a *ratio* compounded of

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the *ratios* of LN to NO and NR to OP; that is, of the *ratios* A to B, C to D, and E to F. Therefore a magnitude of the ſame kind with A and B, which has to B the *ratio* compounded of theſe *ratios* is expreſſed by $A + A.\frac{C-D}{D} + A.\frac{E-F}{F} + A.\frac{C-D}{D}.\frac{E-F}{F}$.

Again, if NR, OP, be ſuppoſed to repreſent G, H, reſpectively, and KV a fourth proportional to OP, KN, and QR; VQ will be equal to KR (14. E. 6.) and conſequently VN will have to NO a *ratio* compounded of the *ratios* of KN to NO and NR to OP; that is, of the *ratios* A to B, C to D, E to F, G to H. But VK is by conſtruction equal to $A.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{G-H}{H} + A.\frac{E-F}{F}.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{E-F}{F}.\frac{G-H}{H}$. And this added to KN above found gives $A + A.\frac{C-D}{D} + A.\frac{E-F}{F} + A.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{E-F}{F} + A.\frac{C-D}{D}.\frac{G-H}{H} + A.\frac{E-F}{F}.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{E-F}{F}.\frac{G-H}{H}$, for the magnitude of the ſame kind with A and B, which has to B the *ratio* compounded of the *ratios* A to B, C to D, E to F, G to H; whence the law of continuation is manifeſt.

The ſame concluſions may be derived from (E. 5.); ſo that no principle can be ſimpler or more geometrical than that here made uſe of.

Thus then theſe magnitudes will ſtand.

1. $A + A.\frac{C-D}{D}$, when two *ratios* are compounded.

2. $A + A.\frac{C-D}{D} + A.\frac{E-F}{F} + A.\frac{C-D}{D}.\frac{E-F}{F}$, when three are compounded.

3. $A + A.\frac{C-D}{D} + A.\frac{E-F}{F} + A.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{G-F}{F} + A.\frac{C-D}{D}.\frac{G-H}{H} + A.\frac{E-F}{F}.\frac{G-H}{H} + A.\frac{C-D}{D}.\frac{E-F}{F}.\frac{G-H}{H}$, when four *ratios* are compounded, &c. &c.

By continuing this operation much farther, I found upon examination that the number of terms in which A is connected with the differences C−D, E−F, G−H, &c. taken one by one, two by two, three by three, &c. if p denote the number of *ratios* compounded, is expreſſed reſpectively by $\frac{p-1}{1}$, $\frac{p-1}{1}.\frac{p-2}{2}$, $\frac{p-1}{1}.\frac{p-2}{2}.\frac{p-3}{3}$, &c. Thus if the *ratio* of A to B be ſuppoſed equal to the *ratios* of C to D, E to F, G to H, &c. reſpectively, theſe expreſſions will give the following ones.

1. $A + \frac{2-1}{1}.A.\frac{A-B}{B}$.

2. $A + \frac{3-1}{1}.A.\frac{A-B}{B} + \frac{3-1}{1}.\frac{3-2}{2}.A.\frac{\overline{A-B}|^2}{B}$.

3. $A + \frac{4-1}{1}.A.\frac{A-B}{B} + \frac{4-1}{1}.\frac{4-2}{2}.A.\frac{\overline{A-B}|^2}{B} + \frac{4-1}{1}.\frac{4-2}{2}.\frac{4-3}{3}$ $A.\frac{\overline{A-B}|^3}{B}$; for magnitudes of the ſame kind with A and B, which have to B reſpectively the duplicate, triplicate, and quadruplicate *ratio* of A to B; where p is ſucceſſively equal to 2, 3, and 4. And univerſally, by the ſame geometrical reaſoning, it is found, that $A + \frac{p-1}{1}.A.\frac{A-B}{B} + \frac{p-1}{1}.\frac{p-2}{2}.A.\frac{\overline{A-B}|^2}{B}$ + &c. $A.\frac{\overline{A-B}|^{p-1}}{B}$ has to B ſuch

B ſuch a multiplicate *ratio* of A to B as is expreſſed by the number p.

In the reaſoning above I fixed on B as the magnitude to which the reſt were to be referred; but I might as well have fixed on A or any of the other magnitudes. Thus, for inſtance, $B + \frac{p-1}{1} . B . \frac{B-A}{A} + \frac{p-1}{1} . \frac{p-2}{2} . B . \frac{\overline{B-A}^{2}}{A} +$ &c. $B . \frac{\overline{B-A}^{p-1}}{A}$ has to A ſuch a multiplicate *ratio* of B to A as is expreſſed by the number p; or A has to $B + \frac{p-1}{1} . B . \frac{B-A}{A} + \frac{p-1}{1} . \frac{p-2}{2} . B . \frac{\overline{B-A}^{2}}{A} +$ &c. $B . \frac{\overline{B-A}^{p-1}}{A}$ the *ratio* of $A + \frac{p-1}{1} . A . \frac{A-B}{B} + \frac{p-1}{1} . \frac{p-2}{2} . A . \frac{\overline{A-B}^{2}}{B} +$ &c. $A . \frac{\overline{A-B}^{p-1}}{B}$ to B; that is, ſuch a multiplicate *ratio* of A to B as is expreſſed by the number p. Each of theſe, indeed, I demonſtrated ſeparately from the ſame ſort of geometrical reaſoning; but for the ſake of brevity I omit ſetting down theſe ſeparate demonſtrations, as they are both contained in general reaſoning above, which furniſhes likewiſe a great variety of other expreſſions, according as certain numbers of the *ratios* C to D, E to F, G to H, &c. are ſuppoſed to be reſpectively equal to, greater or leſs than, the *ratio* of A to B.